Lesson 4. Modeling with Markov Chains

1 Overview

- To specify a Markov chain model, we need to describe
  (1) the state space and the meaning of each state in the setting's context
  (2) the meaning of one time step in the setting's context
  (3) the transition probabilities and initial state probabilities

2 Examples

Example 1. You have been put in charge of inventory management at the Poisson Fish Market. The inventory system for tuna works as follows:

1. Observe the number of crates of tuna in inventory at the beginning of the day. Call this number k. The storage area for tuna can hold at most 4 crates.
2. If $k \in \{0, 1\}$, then order $4 - k$ crates. If $k \in \{2, 3, 4\}$, then order 0 crates. Orders are delivered immediately.
3. Throughout the day, some of these crates of tuna are sold. With probability 1/3, 0 crates are sold. With probability 1/2, 1 crate is sold. With probability 1/6, 2 crates are sold.
4. Observe the number of crates in inventory at the beginning of the next day.

When you start observing the system, there are 2 crates of tuna in inventory. Model this setting as a Markov chain.

Example 2. Consider an experiment involving two urns labeled A and B, containing a total of 5 white balls and 5 red balls among them. You start the experiment by picking one ball at random from each urn in an independent fashion and swapping them. You repeat this process, each time performing this process independently from past occurrences. Suppose initially that urn A has all 5 white balls, and urn B has all 5 red balls. Model this setting as a Markov chain.

- The model in Example 2 is known as the **Bernoulli-Laplace model** for the diffusion of two incompressible gases between two containers, as well as the **Moran model** for the spread of a neutral genetic mutation throughout a population

3 Exercises

Problem 1 (SMAS Exercise 6.17)**.** An automated guided vehicle (AGV) transports parts between four locations: a release station A, machining station B, machining station C, and an output buffer D. The movement of the AGV can be described as making trips from location to location based on requests to move parts. More specifically:

- If the AGV is at the output buffer, it is equally likely to move next to any of the other three locations, A, B, C.
- If the AGV is at the release station, it is equally likely to move next to machining station B or C.
- If the AGV is waiting at either of the machining stations, it is equally likely to move to any of the other three locations.

Suppose at the beginning of the day, the AGV is equally likely to be at any of the four stations. Model this setting as a Markov chain.

Problem 2 (SMAS Exercise 6.27)**.** Banach Bank if studying the use of its automated teller machines (ATM). They have observed the following customer behavior. After inserting a card into an ATM, a customer may perform three types of transactions: deposit, withdrawal, and obtain account information. The bank believes that 50% of all customers start with a withdrawal, 40% start with a deposit, and the remainder start by requesting account information. After completing a transaction, 90% of the customers complete their business; those who do not complete their business are equally likely to select one of the other two types of transactions (for example, if they just made a withdrawal and they do not complete their business, then they are equally likely next to select a deposit or request account information). This pattern continues until their business finally is completed. Model this setting as a Markov chain.

Problem 3. Bit Bucket Computers specializes in installing and maintaining computer systems. Unfortunately, the reliability of their computer systems is somewhat questionable. One of its standard configurations is to install two computers. When both are working at the start of business, there is a 30% chance that one will fail by the close of business, and a 10% chance that both will fail. If only one computer is working at the start of business, then there is a 20% chance it will fail by the close of business. Computers that fail during the day are picked up the following morning, repaired, and then returned the next day after that, before the start of business. Suppose we start observing an office with this configuration on a day with two working computers at the start of business. Model this setting as a Markov chain.